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On the error estimation and T-stability of the Mann iteration

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ABSTRACT

A formula of error estimation of Mann iteration is given in the case of strongly demicontractive mappings. Based on this estimation, a condition of strong convergence is obtained for the same class of mappings. T-stability for a particular case of strongly demicontractive mappings is proved. Some numerical examples showing the accuracy of the proposed estimations are also given.

1. Introduction

The condition of demicontractivity and some smoothness properties (for instance, the demi-closedness at zero) ensure the weak convergence of the Mann iteration, $x_{n+1} = (1 - t_n)x_n + t_n T x_n$, where T is a self mapping on $\mathcal{C}$ (usually $\mathcal{C}$ is a subset of a Hilbert or Banach space), and $\{t_n\}$ is the control sequence [1,2]. To get strong convergence, some additional conditions are needed. In [3] an example is given of a contraction defined on a bounded closed convex subset of a Hilbert space for which the Krasnoselski iteration (a particular case of the Mann iteration) does not converge.

The problem of finding additional conditions for strong convergence was discussed in several papers, including the papers in which the concept of demicontractivity was introduced [1,2]. For example in [1] it is required, as additional condition, that I–T maps closed bounded subsets of C into closed subsets of C (in particular, this is satisfied if T is demicompact). In [2] the existence of a nonzero solution $h \in \mathcal{H}, \ h \neq 0$, of the variational inequality $\langle x - Tx, h\rangle \leq 0, \ \forall x \in \mathcal{C}$ is required. It is obvious that the existence of a nonzero solution of this variational inequality occurs only in very particular cases; an example where this holds, in case of linear equations, is given in [2,4].

This problem is still being studied. In a relative recent paper it is required (as main additional condition) that the mapping T to be *demicompact* [5]. Note that this result was proved in [6] for a strictly pseudocontractive mapping (such mappings are more restrictive than demicontractive ones). The same type of conditions (T is demicompact or $\mathcal{C}$ is a compact subset of $\mathcal{H}$) was proposed in [7]. In [8] the α-demicontractivity is required as an additional condition (a mapping T is said to be α-demicontractive if $\langle x - Tx, x - \alpha p\rangle \geq \lambda \|x - Tx\|^2, \ \forall x \in C, \ p \in \mathrm{Fix}(T)$ for some $\alpha > 1$). Note that such a requirement (to satisfy both conditions, demicontractivity and α-demicontractivity) is very strong. In [8] this requirement is illustrated for real functions.

The problem of strong convergence is closely connected with the problem of error estimation (the estimation of $\|x_n - p\|$, where $\{x_n\}$ is a sequence approximating a fixed point p). There exist several results for Mann iteration (or variants) and for

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some contractive type mappings. For example, in [9] is given the following error estimation of the Mann iteration and for a monotone, nonexpansive mapping:

$$\|x_{n+1} - p\| \le \frac{1}{\sqrt{n+1}} \|x_1 - p\|.$$

For a Lipschitzian strictly pseudocontractive mapping the following estimation is provided in [10]:

$$\|x_{n+1} - p\| \le \rho^n \|x_1 - p\|,$$

where $\rho = 1 - k^2/[4(3+3L+l^2)]$, and $L, \ k$ are the constants from Lipschitzian and strictly pseudocontractivity definitions.

In [11] the error estimation for *Zamfirescu* mappings is given by the formula:

$$\|x_{n+1} - p\| \le \prod_{k=1}^{n} [1 - \alpha_k(1-\delta)] \|x_1 - p\|,$$

where $\{\alpha_n\}$ is control sequence and δ is the constant appearing in *Zamfirescu* type contraction.

More recent results on error estimation for Mann iteration have been reported in [12–14].

The problem of error estimation was less approached for demicontractive class (no results so far, in our knowledge). The contribution of this paper is to find such estimates of the Mann iteration for the case of strong demicontractive mappings. Inequality (3.4) (Theorem 1) provides an a posteriori error estimate depending on the strong demicontractive constants. Based on this estimate a new additional condition is obtained for strong convergence.

The rest of the paper is organized as follows. In Section 2 are discussed some specific lemmas on which strong convergence is usually based. Our main results concerning the error estimation and the strong convergence are presented in Section 3. Section 4 is devoted to prove the T-stability for a particular case of strongly demicontractive mappings. Appendix presents some examples which show the accuracy of the approximation given by inequality (2.6) of Lemma 2, on which our estimation is based, and by inequality (3.4) of Theorem 1.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and let $\| \cdot \|, \langle \cdot \rangle$ denote the norm and scalar product on $\mathcal{H}$, respectively. Let C be a closed convex subset of $\mathcal{H}$ and let T be a mapping of C into itself, $T : C \to C$, and let $\mathrm{Fix}(T)$ denote the set of fixed points of T.

Definition 1. The mapping $T : C \to C$ is said to be demicontractive if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \le \|x - p\|^2 + K\|x - Tx\|^2, \quad \forall (x, p) \in C \times \mathrm{Fix}(T), \tag{2.1}$$

where $K \ge 0$.

Remark 1. The *demicontractive* class of mappings was used in [15] to study the Mann-type iteration in finite dimensional spaces. This concept was further extended to Hilbert spaces in [1,2]; the term *demicontractivity* was proposed by Hiks and Kubicek [1].

In the following we strengthen the notion of demicontractivity; instead of (2.1), we consider the condition

$$\|Tx - p\|^2 \le a\|x - p\|^2 + K\|x - Tx\|^2, \quad \forall (x, p) \in \mathcal{C} \times \mathrm{Fix}(T), \tag{2.1$'$}$$

where $a \in (0, 1)$ and $K \ge 0$. We will say that T is *strongly demicontractive*.

If T is strongly demicontractive (or T has the (L, b)-property, see below) then the fixed point is unique. Indeed, if q is another fixed point of T, then

$$\|q - p\|^2 = \|Tq - p\|^2 \le a\|q - p\|^2 + K\|Tq - q\|^2 = a\|q - p\|^2,$$

which implies $a \ge 1$, contrary to the hypothesis.

Remark 2. The condition (2.1′) is mentioned also in [16] but with different conditions on a and K (more precisely, $a\ , K > 1$). This class of mappings is called *firmly pseudo-demicontractive*.

It is worthwhile to mention that the concept of strong demicontractivity is similar to that of (L, b)-*property* [17] which is defined on a metric space X by

$$d(T(x), p) \le bd(x, p) + Ld(x, T(x)), \quad \forall (x, p) \in X \times \mathrm{Fix}(T),$$

where $b \in (0, 1)$ and $L \ge 0$. The similarity between strong demicontractivity and (L, b)-property is obvious.

Note that a condition of the same type was introduced by Berinde [12], more precisely:

$$d(T(x), Ty) \le \delta d(x, y) + Ld(y, T(x)), \quad \forall x, y \in X, \tag{B}$$

and he called *almost contractive* a mapping satisfying (B). Berinde [12] proved the following theorem.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a weak contraction, i.e., a mapping satisfying (B) for some $\delta \in (0, 1)$ and some $L \geq 0$. Then T has a unique fixed point p and the Picard iteration converges to p for any $x_0 \in X$. The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \ldots,$$
$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 0, 1, 2, \ldots,$$

hold, where δ is the constant appearing in (B).

Remark 3. The error estimations given by Theorem B are very similar with that given by the well known Banach contraction principle. Observe that these error estimations depend only by the constant δ.

Most results on the strong convergence of the Mann iteration (or its variants) are based on a simple lemma concerning the behavior of a numerical sequence $\{d_n\}$ which satisfies an inequality of the form

$$d_{n+1} \leq \alpha_n d_n + \varepsilon_n, \tag{2.2}$$

where α_n and ε_n satisfy appropriate conditions.

Our contribution is based on the following two lemmas. The first lemma gives conditions for the convergence of the sequence $\{d_n\}$ defined by (2.2); this lemma is commonly used in the proof of convergence and stability of iterative procedures (often it is the key of the proof); the second lemma completes the assertion of Lemma 1, giving an estimation of error.

Lemma 1. *Let $\{d_n\}$, $\{\varepsilon_n\}$ be nonnegative sequences of real numbers satisfying $d_{n+1} \leq \alpha d_n + \varepsilon_n$, for all $n \in \mathbb{N}$, $0 \leq \alpha < 1$ and $\lim \varepsilon_n = 0$. Then, $\lim d_n = 0$.*

In [18] several variants of this lemma are presented.

Lemma 2. *Let $\{d_n\}$ be a nonnegative sequence satisfying*

$$d_{n+1} \leq \alpha d_n + \beta \varepsilon_n, \tag{2.3}$$

where $0 < \alpha < 1$, $\beta > 0$ and $\{\varepsilon_n\}$ is a nonnegative sequence that satisfies the condition

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} \geq 2\alpha, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Then

$$d_{n+1} \leq d_0 \alpha^{n+1} + \beta(2\alpha\varepsilon_{n-1} + \varepsilon_n). \tag{2.5}$$

Proof. Using the inequality (2.3), we obtain

$$d_{n+1} \leq d_0 \alpha^{n+1} + \beta \sum_{i=0}^{n} \varepsilon_i \alpha^{n-i}.$$

Then it is simple to prove by induction on n that

$$\sum_{i=0}^{n} \varepsilon_i \alpha^{n-1} \leq 2\alpha\varepsilon_{n-1} + \varepsilon_n. \quad \square \tag{2.6}$$

Remark 4. If $\varepsilon_n \to 0$ then from (2.5) it results $d_n \to 0$.

Remark 5. The key of the proof of Lemma 2 is the inequality (2.6). Usually ε is a function, $\varepsilon : \mathbb{R} \to \mathbb{R}$ and $\varepsilon_n = \varepsilon(x_n)$. The inequality (2.6) gives a superior estimation of the sum in the left hand of (2.6), which allows us to deduce the convergence to zero of $\{d_n\}$. It is worth mentioning that the estimation depends on α and on the last two values of ε. The accuracy of this estimation in the case of two particular functions ε is shown in the Appendix.

Remark 6. It is obvious that if $\{\varepsilon_n\}$ satisfies the condition (2.4) and $\varepsilon_n \to 0$, then the convergence rate of the sequence $\{\varepsilon_n\}$ is linear.

3. Error estimation

The strong demicontractivity is equivalent with

$$\langle x - Tx, x - p\rangle \geq \frac{1-a}{2}\|x-p\|^2 + \frac{1-K}{2}\|Tx-x\|^2, \quad \forall (x,p) \in C \times \mathrm{Fix}(T). \tag{3.1}$$

This results immediately from $\|Tx-p\|^2 - a\|x-p\|^2 - K\|Tx-x\|^2 = (1-a)\|x-p\|^2 + (1-K)\|Tx-x\|^2 - 2\langle x-Tx, x-p\rangle$. We will use the following notations: $\varepsilon(x) := \|Tx-x\|^2$ and $\varepsilon_n := \varepsilon(x_n)$, $T_t := (1-t)I + tT$, where I is the identity mapping. Let f be a quadratic polynomial defined by $f(t) = t^2 - (1-K)t$.

Theorem 1. *Let T be a strongly demicontractive mapping, $0 < a < 1$, $K \geq 0$. Assume that there exist positive numbers $\theta_1, \ \theta_2, 0 < \theta_1 \leq \theta_2 < \min\{1, \ 1-(1-a)(1-K)\}$ such that*

$$\frac{\|TT_t x - T_t x\|^2}{\|Tx-x\|^2} \geq 2\theta_2, \tag{3.2}$$

for all $x \in C$ and for t satisfying

$$\frac{1-\theta_2}{1-a} \leq t_n \leq \frac{1-\theta_1}{1-a}. \tag{3.3}$$

Let $\{x_n\}$ be the sequence generated by Mann iteration with the control sequence verifying (3.3) and suppose that $\{x_n\} \subset C$. Then the following a posteriori error estimation for the Mann sequence $\{x_n\}$ holds

$$\|x_{n+1}-p\|^2 \leq \|x_0-p\|^2\theta_2^{n+1} + M(2\theta_2\varepsilon_{n-1} + \varepsilon_n), \tag{3.4}$$

where $M = f(\frac{1-\theta_1}{1-a})$.

Proof. Using the Mann iteration formula and the inequality (3.1), we obtain

$$\|x_{n+1}-p\|^2 \leq [1-t_n(1-a)]\|x_n-p\|^2 + [t_n^2 - (1-K)t_n]\|Tx_n - x_n\|^2.$$

From (3.3) we have $0 < \theta_1 \leq 1 - t_n(1-a) \leq \theta_2 < 1$. The function f is increasing on $[\max\{0, 1-K\}, \ \infty)$. Since $\theta_2 \leq 1-(1-a)(1-K)$, it follows that $1-K \leq \frac{1-\theta_2}{1-a} \leq t_n$. Thus

$$f(t_n) = t_n^2 - (1-K)t_n \leq f\left(\frac{1-\theta_1}{1-a}\right) = M.$$

Hence

$$\|x_{n+1}-p\|^2 \leq \theta_2\|x_n-p\|^2 + M\varepsilon_n.$$

The sequence $\{\|x_n-p\|^2\}$ satisfies the inequality (2.3) with $\alpha = \theta_2$ and $\beta = M$.

Now, from (3.2) it results

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{\|Tx_{n+1}-x_{n+1}\|^2}{\|Tx_n-x_n\|^2} = \frac{\|TT_{t_n}x_n - T_{t_n}x_n\|^2}{\|Tx_n-x_n\|^2} \geq 2\theta_2.$$

Lemma 2 may be applied. □

Corollary 1. *Suppose that T satisfies the conditions of Theorem 1 and that it is asymptotically regular, i.e. $\|T^{(n+1)}x_0 - T^{(n)}x_0\| \to 0$ for some point x_0. Then the sequence $\{x_n\}$ generated by the Mann iteration with control sequence satisfying (3.3) and starting point x_0 converges strongly to the fixed point of T.*

The condition (3.2) is relatively strong. In the sequel we show that if T has some particular properties, then this condition is satisfied. This fact is illustrated below with the help of two examples.

The case of a differentiable mapping $f : \mathbb{R} \to \mathbb{R}$.

Suppose that f is strongly demicontractive, with $0 < a < 1$, $K > 0$ and that the derivative f' of f satisfies $f'(x) \geq m, \ \forall x \in \mathbb{R}$ with $0 < m \leq 1$. We have

$$\begin{aligned} f(f_t(x)) - f_t(x) &= f[x + t(f(x)-x)] - x - t(f(x)-x) \\ &= [1 - t + tf'(\xi)](f(x)-x), \end{aligned}$$

where $\xi = x + \eta t(f(x)-x), \ 0 < \eta < 1$. Let θ_2 be a positive number satisfying

$$\frac{1-\theta_2}{1-a} < \frac{1-\sqrt{2\theta_2}}{1-m}.$$

Observe that such a θ_2 always exists. Let θ_1 be such that $0 < \theta_1 < \theta_2$ and $\frac{1-\theta_1}{1-a} < \frac{1-\sqrt{2\theta_2}}{1-m}$ (if θ_1 is sufficiently close to θ_2 the later condition will be satisfied).

Now, if t satisfies (3.3) (with $\theta_1,\ \theta_2$ defined above) then $1-t+tf'(\xi) \geq 1-t+tm \geq \sqrt{2\theta_2}$ and

$$\frac{|f(f_t(x)) - f_t(x)|^2}{|f(x)-x|^2} \geq 2\theta_2.$$

The example below presents a real function which satisfies the conditions of Theorem 1.

Example 1. Let $C = [0.5, 1.5]$ and let f be a function $f : [0.5, 1.5] \to [-0.5, 1.5]$ defined by

$$f(x) = \begin{cases} -x+2-(x-1)^4 & \text{if } x < 0, \\ -x+2+(x-1)^3 & \text{if } x \geq 0. \end{cases}$$

This function is demicontractive with $a = 0.15,\ K = 0.25$. If we take $\theta_1 = 0.16,\ \theta_2 = 0.23$ then $t \in [0.906, 0.988]$ and $\|TT_tx - T_tx\|^2/\|Tx-x\|^2 \leq 0.46,\ \forall x \in [0.5, 1.5]$. It can be seen that all conditions of Theorem 1 are satisfied.

The true error and the error estimation for x_{50} (with starting point $x_0 = 0.5$) are, respectively:

The true error $:= 4.79 \times 10^{-6}$,

The error estimation $:= 5.628 \times 10^{-6}$.

Note that the first term in (3.2), $\|x_0 - p\|^2\theta_2^{51}$, is, practically, zero.

The case of a linear mapping and Picard iteration.

Let T be a linear mapping, $Tx = Ax + b$, where A is an $m \times m$ matrix, $b \in \mathbb{R}^m$. As usual p will denote the fixed point of T. Consider the particular case of Mann iteration with constant control sequence, $t_n = 1,\ \forall n \in \mathcal{N}$, i.e. the Picard iteration, $x_{k+1} = Tx_k$. We will suppose that both A and $A - I$ are invertible. Let c_M denote the condition number of a matrix M.

Let us observe that in this case T is a strongly demicontractive mapping; indeed, for any $0 < a < 1$ if we take $K \geq (\|A\|^2 - a)\|(A-I)^{-1}\|^2$ then (2.1′) is satisfied.

By a simple computation we obtain

$$\frac{\|TT_tx - T_tx\|^2}{\|Tx-x\|^2} = \frac{\|TTx - Tx\|^2}{\|Tx-x\|^2} \geq \|A^{-1}\|^{-2}c_{A-I}^{-2}.$$

We can take $a = \min\{1, \|A^{-1}\|^{-2}c_{A-I}^{-2}/2\}$, and K given above. Then we choice the positive numbers θ_1 and θ_2 as $\theta_1 = \theta_2 = a$ and then $t_n = 1$. The conditions of Theorem 1 are satisfied and we can use (3.4) for error estimation.

Fig. 1(c) shows the accuracy of this estimation; on the same plot is drawn the well known error estimation for α-contractions.

4. *T*-stability

Let $\{x_n\}$ be a sequence given by the iteration procedure $x_{n+1} = G(n, x_n)$, where $G : \mathbb{N} \times \mathcal{C} \to \mathcal{C}$. For example, the Mann iteration is defined with the help of control sequence $\{t_n\}$ and $G(n, x) = (1-t_n)x + t_nTx$. The set of fixed points of G is $\mathrm{Fix}(G) = \{p \mid G(n, p) = p,\ \forall n \in \mathbb{N}\}$. Suppose that $\mathrm{Fix}(G) \neq \emptyset$ and that the sequence $\{x_n\}$ generated by the iteration procedure $x_{n+1} = G(n, x_n)$ converges to some fixed point p. Let $\{y_n\}$ be any sequence in $\mathcal{C}$ and let us define $\varepsilon_n = \|y_{n+1} - G(n, y_n)\|$. If $\varepsilon_n \to 0$ implies that $y_n \to p$, then the iteration procedure $x_{n+1} = G(n, x_n)$ is said to be T-stable.

The T-stability of Picard iteration for α-contractions was shown already by Ostrowski [19]. Harder and Hicks [20] established T-stability results of Picard, Mann and Kirk iterations for Zamfirescu contractions (in the case of Picard and Mann iterations) and for α-contraction (in the case of Kirk iteration). Osilike [21] generalized the results of Harder and Hicks for a mapping T satisfying the following condition

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx),$$

where $a \in [0, 1)$ and $L \geq 0$. More recently, in [22] it is shown the T-stability of Picard iteration for mapping having (L, b)-property.

The theorem below shows that the Mann iteration is T-stable for some demicontractive mappings and for suitable control sequence.

Theorem 2. *Suppose that $T : \mathcal{C} \to \mathcal{C}$ is strongly demicontractive with the constants $0 < a < 1,\ 0 \leq K < 1$. Assume furthermore that*

$$a + \frac{4K}{(1-K)^2} \leq 1, \tag{4.1}$$

and that the control sequence satisfies $\tau \leq t_n \leq 1$ for some $\tau > 0$ Then the Mann iteration is T-stable.

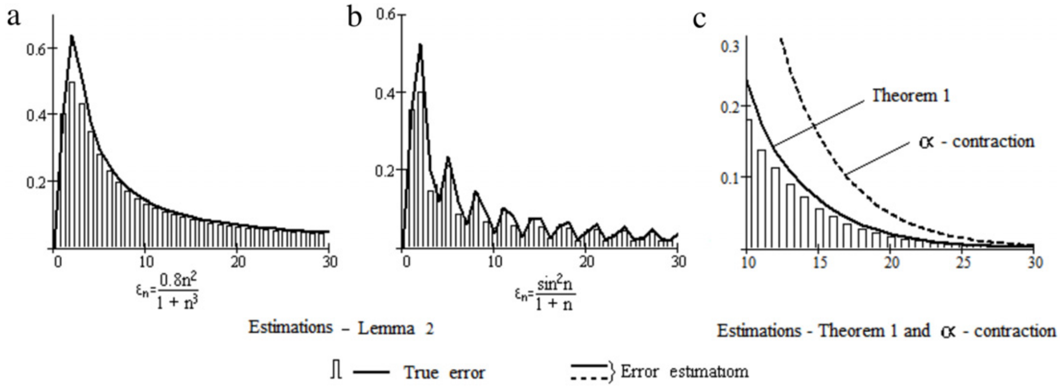

Fig. 1. Illustration of error estimation accuracy based on (2.5) and (3.4). Bars represent the true error and the continuous line/interrupted line represents the error estimations.

Proof. Using (3.1) we have

$$\|x - Tx\| \leq \frac{2}{1-K}\|x - p\|,$$

and substituting this in (2.1′) it results

$$\|Tx - p\| \leq \sqrt{a + \frac{4K}{(1-K)^2}}\,\|x - p\|. \tag{4.2}$$

Let $\{y_n\}$ be a sequence in $\mathcal{C}$ and define $\varepsilon_n = \|y_{n+1} - (1-t_n)y_n - t_nTy_n\|$. We have

$$\begin{aligned}\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-t_n)y_n - t_nTy_n\| + \|(1-t_n)y_n - p + t_nTy_n\|\\ &= \|(1-t_n)(y_n - p) + t_n(Ty_n - p)\| + \varepsilon_n\\ &\leq (1-t_n)\|y_n - p\| + t_n\|Ty_n - p\| + \varepsilon_n\\ &\leq \left(1 - t_n + t_n\sqrt{a + \frac{4K}{(1-K)^2}}\right)\|y_n - p\| + \varepsilon_n.\end{aligned}$$

Now, if we take $\alpha := 1 - \tau + \tau\sqrt{a + \frac{4K}{(1-K)^2}}$, then $0 < \alpha < 1$ and

$$1 - t_n + t_n\sqrt{a + \frac{4K}{(1-K)^2}} \leq 1 - \tau + \tau\sqrt{a + \frac{4K}{(1-K)^2}} = \alpha.$$

It results

$$\|y_{n+1} - p\| \leq \alpha\|y_n - p\| + \varepsilon_n.$$

Lemma 1 can be applied and $\varepsilon_n \to 0$ implies $y_n \to p$. □

Remark 7. The condition (4.1) is relatively strong. Indeed, since $a > 0$ this condition forces that $4K/(1-K)^2 < 1$, which, together with $0 \leq K < 1$ gives $0 < K < 3 - \sqrt{8} \approx 0.172$. Thus a strong demicontractive satisfying (4.1), also satisfies (4.2), and it is strictly quasi-nonexpansive.

The example below presents a real function, $f : [-1, 0.4] \to [-1, 0.4]$, which is strictly demicontractive and $a,\ K$ satisfy condition (4.1).

Example 2. Let f be the function

$$f(x) = \begin{cases} 0.5x & \text{if } -1 \leq x \leq 0.15,\\ -1.1x + 0.1 & \text{if } 0.15 < x \leq 0.4.\end{cases}$$

This function is strictly demicontractive with $a = 0.35,\ K = 0.12$ and $a + 4K/(1-K)^2 = 0.97$. Note that f is not contractive.

Appendix

The purpose of the Appendix is to provide examples on the accuracy of the proposed error estimations, inequalities (2.5) and (3.4).

Fig. 1(a) and (b), show the accuracy of the error estimation (2.5) for two sequences $\{\varepsilon_n\}$, $\varepsilon_n = \frac{0.8n^2}{1+n^3}$ and $\varepsilon_n = \frac{\sin^2 n}{1+n}$ and for $a = 0.35$. These sequences were chosen such that $\varepsilon_n \to 0$ and $\varepsilon_{n+1}/\varepsilon_n \geq \mu > 0$ (for the first sequence $\mu = 0.889$ and for the second sequence $\mu = 0.778$) and a was chosen such that $\mu > 2a$.

In Fig. 1(c) is represented the true errors and error estimations for a linear function $T : S \to S$, where S is the unit ball in $\mathbb{R}^4$, and $Tx = Ax$, where A is the matrix with lines: $(0.2, 0.1, -0.4, 0.1)$, $(0.3, 0.3, -0.1, 0.5)$, $(0.6, -0.2, 0.5, 0)$, $(0.1, 0.2, 0.1, 0.5)$. The constants appearing in the demicontractivity condition are $a = 3.653 \times 10^{-5}$, $K = 20.852$. In this figure are plotted two error estimations showing, respectively, the accuracy given by (3.4), and by the well known formula for α-contraction, $\|x_n - p\| \leq \frac{\alpha}{1-\alpha}\|x_n - x_{n-1}\|$ (in this case, $\alpha = 0.893\ldots$).

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